EXERCISES: ENUMERATIVE COMBINATORICS

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1. *In how many different ways can we partition an n-set into two parts if one part has three elements and the other part has all the remaining elements?*

   Solution:
   Consider an n-set X. We want to determine two disjoint subsets, A and B, of X such that $X = A \cup B$, $|A| = 3$ and $|B| = n - 3$. If we fix the elements of A, then B turns out to be the complement of A; in other words $B = X \setminus A$. Thus to count the number of pairs (A, B) that satisfy the above conditions, it is equivalent to count the number of ways in which we can determine a subset of cardinality 3 from an n-set: this value corresponds to the binomial number $\binom{n}{3}$.

2. *There are 4 characters, two letters of the alphabet followed by two digits, which appear in a sequence. Determine the number of sequences if:*

   (a) *all characters can be repeated;*
   (b) *letters cannot be repeated;*
   (c) *no character can be repeated.*

   Solution:
   A sequence formed by two letters followed by two digits can be represented as $a_1\, a_2\, b_1\, b_2$; where a_1, a_2 denote letters, while b_1, b_2 denote digits.

   (a) If all characters can be repeated we can choose each letter in 26 ways and each digit in 10 ways. Thus, by the multiplication principle, the number of sequences is $26 \cdot 26 \cdot 10 \cdot 10$.
   (b) If letters cannot be repeated, we can choose the first letter in 26 ways, while the second one can be chosen only in 25 ways (we cannot use again the letter corresponding to the first character). Moreover each digit can be selected in 10 ways. Thus the number of sequences is $26 \cdot 25 \cdot 10 \cdot 10$.
   (c) Using a similar argument to the preceding one also for digits, we obtain that the number of sequences is $26 \cdot 25 \cdot 10 \cdot 9$.

3. *Find the number of ways to form a four-letter sequence using the letters A, B, C, D, E if*

   (a) *repetitions of letters are permitted*;
   (b) *repetitions are not permitted*;
   (c) *the sequence contains the letter A but repetitions are not permitted*;
   (d) *the sequence contains the letter A but repetitions are permitted*;

   Solution:
   We can represent a four-letter sequence as $a_1a_2a_3a_4$, where $a_i \in \{A, B, C, D, E\}$ for $i = 1, 2, 3, 4$.

(a) Each elements a_i can be any of the available five letters. Thus the number of sequences is 5^4.

(b) Since we cannot repeat characters, a_1 can be chosen in 5 ways, a_2 in 4, a_3 in 3 and a_4 only in 2. Hence the number of sequences is $5 \cdot 4 \cdot 3 \cdot 2$.

(c) Let

$$A_1 = \{Aa_2a_3a_4, \text{where } a_i \in \{B, C, D, E\}, \text{ without repetitions}\}$$

be the set of all sequences without repetitions where A is the first character and the remaining letters belong to the set $\{B, C, D, E\}$. In the same way we can define A_2, A_3 and A_4. The number of sequences that satisfy condition (c) is

$$|A_1 \cup A_2 \cup A_3 \cup A_4|.$$

Since these sets are pairwise disjoint, by the addition principle, the required value is

$$|A_1| + |A_2| + |A_3| + |A_4|.$$

We only need to compute the cardinality of these sets.
Consider A_1. The cardinality of A_1 coincides with the number of sequences of length 3 formed with the letters B, C, D, E with no repetitions, that is $4 \cdot 3 \cdot 2$. Using the same argument we obtain that $|A_2| = |A_3| = |A_4| = 4 \cdot 3 \cdot 2$. Thus

$$|A_1| + |A_2| + |A_3| + |A_4| = 4 \cdot 4 \cdot 3 \cdot 2.$$

Note that is also possible to solve the exercise in the following way. The position of the letter A can be chosen in 4 ways. The remaining three symbols in the word are chosen from the set $\{B, C, D, E\}$, without repeating any letter. Thus the number of sequences is $4 \cdot 4 \cdot 3 \cdot 2$.

(d) We can construct the required sequences in the following way:

- fix the position of A
- select each one of the three remaining symbols from the set $\{A, B, C, D, E\}$

The first event can be done in 4 ways, while the second one in 5^3 ways (each of the remaining symbols can be chosen in 5 ways). Thus, by the multiplication principle, the required value is $4 \cdot 5^3$.

4. *How many subsets of the set* $[10]$ *contain only even integers?*

Solution:
We are interested in computing the number of subsets of $\{1, 2, \ldots, 9, 10\}$ containing only even integers; in other words we want to determine the number of subsets of the set $\{2, 4, 6, 8, 10\}$, which turns out to be 2^5.

5. *Find the number of solutions of* $x_1 + x_2 + \cdots + x_n = r$, *where each variable is either* 0 *or* 1.

Solution:
In order to have a solution of the given equation we have to assign to r variables the value 1 and to the remaining ones the value 0. In other words, if we consider the set $\{x_1, x_2, \ldots, x_n\}$, we have to choose a subset of cardinality r, that corresponds to the variables equal to 1, while the remaining variables will be equal to 0. Such a subset can be chosen in $\binom{n}{r}$ ways.

6. *There are* 65 *students in a class. Among them a total of* 41 *can play piano,* 20 *the violin,* 10 *the guitar,* 10 *both piano and violin,* 7 *piano and guitar,* 5 *violin and guitar but only* 2 *can play all three instruments. How many students of the class cannot play any of them?*

   Solution:
   Let I be the set of students, A_1 the set of students that play piano, A_2 the set of students that play violin and A_3 the set of students that play guitar. We want to compute the number of students that cannot play any of the three instruments, in other words $|A_1' \cap A_2' \cap A_3'|$, where A_i' is the complement of A_i, $i = 1, 2, 3$. To solve the exercise we use the principle of inclusion-exclusion: from Sylvesters's identity we obtain that

$$|A_1' \cap A_2' \cap A_3'| = |I| - s_1 + s_2 - s_3,$$

   where $s_1 = \sum |A_i|$, $s_2 = \sum |A_i \cap A_j|$ and $s_3 = |A_1 \cap A_2 \cap A_3|$.
   In our case we get

$$|I| = 65,$$
$$s_1 = 41 + 20 + 10 = 71,$$
$$s_2 = 10 + 7 + 5 = 22,$$
$$s_3 = 2;$$

   and so

$$|A_1' \cap A_2' \cap A_3'| = 14.$$

7. *How many subsets of the set* $[10]$ *contain at least one odd integer?*

   Solution:
   Let I be the set of subsets of $[10]$ and denote

$$\begin{aligned} A_1 &= \text{subsets that contain } 1, \\ A_2 &= \text{subsets that contain } 3, \\ A_3 &= \text{subsets that contain } 5, \\ A_4 &= \text{subsets that contain } 7, \\ A_5 &= \text{subsets that contain } 9. \end{aligned}$$

   We want to compute $|A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5|$. Since these subsets of I have non-empty intersection, we can solve the exercise by using the principle of inclusion-exclusion, in particular Da Silva's formula:

$$|A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5| = s_1 - s_2 + s_3 - s_4 + s_5,$$

   where $s_1 = \sum |A_i|$, $s_2 = \sum |A_i \cap A_j|$, $\ldots$, $s_5 = |A_1 \cap A_2 \cap A_3 \cap A_4 \cap A_5|$.

   We begin by computing the cardinality of A_1. An element of A_1 is a subset of $[10]$ that contains 1; so it can be determined by choosing a subset of $[10]\setminus\{1\}$ and then adding 1. Thus the number of element of A_1 is equal to the number of subsets of $[10] \setminus \{1\}$, which is 2^9. Using the same argument $|A_2| = |A_3| = |A_4| = |A_5| = 2^9$ and so $s_1 = 5 \cdot 2^9 = \binom{5}{1} \cdot 2^9$.

We now compute the value of s_2. Let us start with $|A_1 \cap A_2|$. An element of $A_1 \cap A_2$ is a subset of $[10]$ that contains 1 and 3; so it can be determined by choosing a subset of $[10] \setminus \{1,3\}$ and then adding 1 and 3. Thus the number of element of $A_1 \cap A_2$ is equal to the number of subsets of $[10] \setminus \{1,3\}$, which is 2^8. Using the same argument it is easy to prove that every term in the sum s_2 is equal to 2^8. Since in the sum s_2 there are $\binom{5}{2}$ values (a term in the sum s_2 is determined by choosing two subsets among A_1, A_2, A_3, A_4, A_5), we get that $s_2 = \binom{5}{2} \cdot 2^8$.

Repeating the same procedure for the remaining values we obtain that

$$s_3 = \binom{5}{3} \cdot 2^7,$$

$$s_4 = \binom{5}{4} \cdot 2^6,$$

$$s_5 = \binom{5}{5} \cdot 2^5 = 2^5.$$

Hence

$$|A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5| = \binom{5}{1} \cdot 2^9 - \binom{5}{2} \cdot 2^8 + \binom{5}{3} \cdot 2^7 - \binom{5}{4} \cdot 2^6 + \binom{5}{5} \cdot 2^5.$$

Alternatively, we could make an indirect count by observing that the number of subsets of $[10]$ that contain at least one odd integer can be obtained by subtracting from the number of subsets of $[10]$ (2^{10}), the number of subsets formed by only even integers (2^5). Hence $2^{10} - 2^5$ is the required value.

8. *A byte is a binary sequence of length* 8.

   (a) *Find the number of bytes.*

   (b) *Find the number of bytes that begin with* 10 *and end with* 01.

   (c) *Find the number of bytes that begin with* 10 *but do not end with* 01.

   (d) *Find the number of bytes that begin with* 10 *or end with* 01.

Solution:
A byte is a binary sequence of length 8, so it can be represented as $x_1 x_2 \cdots x_8$, where $x_i \in \{0,1\}$ for $i = 1, 2, \ldots, 8$.

(a) Each elements x_i can be 0 or 1; so the number of bytes is 2^8.

(b) We want to find the number of bytes that begin with 10 and end with 01; so we want to compute the number of binary sequences of type $1\,0\,x_3 x_4 x_5 x_6\,0\,1$, where $x_i \in \{0,1\}$ for $i = 3, 4, 5, 6$. The number of such sequences is 2^4.

(c) The number of bytes that begin with 10 but do not end with 01 can be obtained by subtracting from the number of bytes that begin with 10 (2^6), the number of bytes that begin with 10 and end with 01 (2^4). Hence $2^6 - 2^4$ is the required value.

(d) Let

$$A_1 = \text{set of bytes that begin with 10},$$
$$A_2 = \text{set of bytes that end with 01}.$$

The number of bytes that begin with 10 or end with 01 is

$$|A_1 \cup A_2|.$$

Since A_1 and A_2 have non-empty intersection, we can solve the exercise by using the principle of inclusion-exclusion, in particular Da Silva's formula:

$$|A_1 \cup A_2| = s_1 - s_2,$$

where $s_1 = \sum |A_i|$ and $s_2 = |A_1 \cap A_2|$. It is easy to see that

$$s_1 = |A_1| + |A_2| = 2^6 + 2^6 = 2 \cdot 2^6$$

and

$$s_2 = |A_1 \cap A_2| = 2^4.$$

Hence

$$|A_1 \cup A_2| = 2 \cdot 2^6 - 2^4 = 2^7 - 2^4.$$

9. *Find the number of ways a mother can distribute* 9 *identical sweets to her three children so that each child gets at least* 2 *sweets.*

Solution:
Consider three children, say A, B and C. Each one of them has to receive at least two sweets, which means that there are 3 remaining sweets that have to be assigned to A, B, C.
A possible assignment can be represented by putting three dots in a row, symbolizing the 3 remaining sweets, and inserting two dividers in the following way: the number of dots before the first divider correspond to sweets for A, the number of dots between the two dividers correspond to sweets for B and the number of dots after the second divider correspond to sweets for C.
For example, the configuration $\bullet|\bullet|\bullet$ represents the assignment in which A, B and C get 1 sweet each (i.e. each child has a total of 3 sweets); the configuration $|\bullet\bullet\bullet|$ is the assignment in which B gets all the remaining sweets (i.e. A and C have a total of 2 sweets each, whereas B has a total of 5 sweets).
In other words a possible configuration is a sequence of length 5 formed by 3 dots and 2 dividers: if we fix the positions of the dividers, the remaining elements have to be dots. Thus we have a 5-set corresponding to the available positions and we have to select a subset of cardinality 2 that corresponds to the positions of the dividers: this can be done in $\binom{5}{2} = 10$ ways.

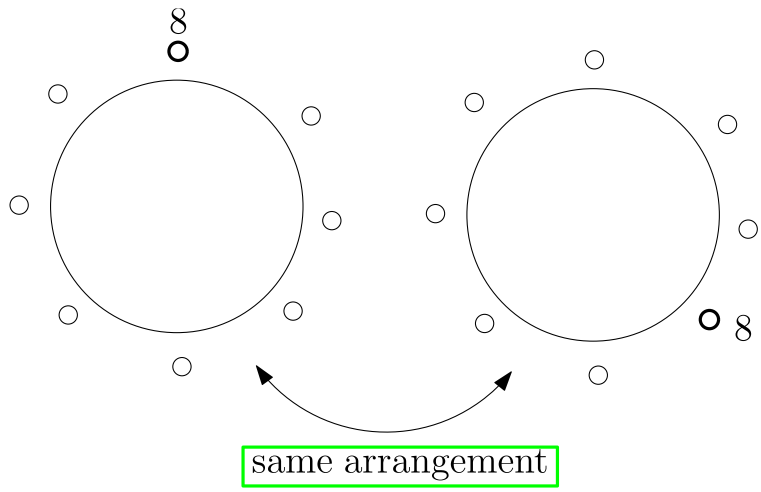
Note that we can also solve the exercise by listing all the possible configurations:

$$||\bullet\bullet\bullet \qquad |\bullet|\bullet\bullet \qquad |\bullet\bullet|\bullet \qquad |\bullet\bullet\bullet| \qquad \bullet||\bullet\bullet$$

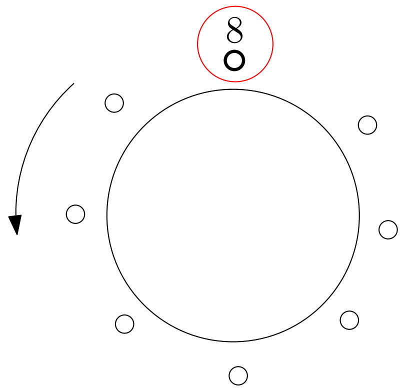
$$\bullet|\bullet|\bullet \qquad \bullet|\bullet\bullet| \qquad \bullet\bullet||\bullet \qquad \bullet\bullet|\bullet| \qquad \bullet\bullet\bullet||$$

10. *Eight people are to be seated around a large round table. Find the number of possible seating arrangements.*

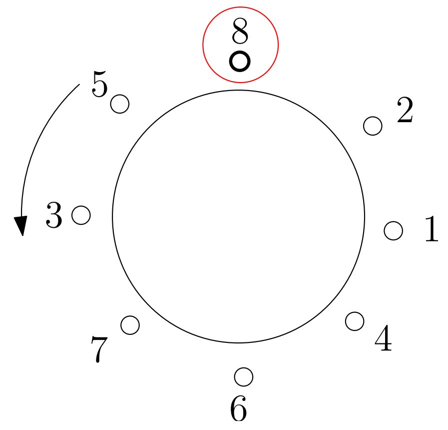
Solution:
Denote the eight people as $\{1, 2, 3, 4, 5, 6, 7, 8\}$.
Starting with the empty table, we see that person number 8 can select its position in only 1 way because the round table is empty and it can be rotated.



Given a possible arrangement, consider the position of person number 8 as a "starting point" and read the positions of the remaining people starting from 8 and in counterclockwise order.



With this procedure we obtain an ordered sequence of length 7 formed by the elements $1, 2, 3, 4, 5, 6, 7$ that identifies the given arrangement.
For example the following arrangement is completely determined by the ordered sequence 5 3 7 6 4 1 2.



Moreover, given a permutation of $1, 2, 3, 4, 5, 6, 7$, we obtain a possible arrangement. In other words the number of possible seating arrangements of 8 people around a round table is equal to the number of permutations of the elements $1, 2, 3, 4, 5, 6, 7$, which turns out to be $7!$.

11. *Use the characteristic equation to find a formula for u_n, when the sequence (u_n) is defined by*

$$u_{n+2} = 3u_{n+1} + 4u_n, \ (n \geq 0)$$

where $u_0 = 1$, $u_1 = 3$.

Solution:
The characteristic equation is $x^2 - 3x - 4 = 0$ whose roots are -1 and 4. Thus any general solution of the given recurrence relation is of the form $u_n = A \cdot 4^n + B \cdot (-1)^n$, where A, B are constants to be evaluated using the two given initial conditions. If $u_0 = 1$, then $A + B = 1$. If $u_1 = 3$, then $4A - B = 3$. Solving these two simultaneous equations in A and B we get $A = \frac{4}{5}$ and $B = \frac{1}{5}$, giving the unique solution to the problem $u_n = \frac{4}{5} \cdot 4^n + \frac{1}{5} \cdot (-1)^n = \frac{1}{5} \cdot \{4^{n+1} + (-1)^n\}$.