EXERCISES: GRAPH THEORY

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1. *Prove that there is no graph with seven vertices that is regular of degree* 3.

   Solution:
   Given a graph $G = (V, E)$, the sum of the degrees of the vertices of G is equal to twice the number of edges; in other words
   $$\sum_{v \in V} d(v) = 2|E|.$$
   In our case we have that all seven vertices have degree equal to 3 and thus we obtain that
   $$\sum_{v \in V} d(v) = 3 \cdot 7 = 21,$$
   which is an odd number. This implies that there is no graph with seven vertices that is regular of degree 3.

2. *G is a connected planar graph of order 24 and it is regular of degree 3. How many faces are in a planar representation of G?*

   Solution:
   Recall that if G is a connected planar graph with n vertices and m edges, then the number of faces is p, where $n - m + p = 2$ (Euler's formula). In our case $n = 24$. To compute the number of edges of G we use the formula
   $$m = |E| = \frac{1}{2} \sum_{v \in V} d(v).$$
   Since G is a regular graph of degree 3 and order 24, we have that
   $$m = |E| = \frac{1}{2} \sum_{v \in V} d(v) = \frac{1}{2} \cdot 24 \cdot 3 = 36.$$
   Thus $p = 2 - n + m = 2 - 24 + 36 = 14$.

3. *Let n be a positive integer. The n-cube is the graph, denoted Q_n, whose vertices are the 2^n possible length-n strings of 0s and 1s. For example, the vertices of Q_3 are* $000, 001, 010, 011, 100, 101, 110, 111$. *Two vertices of Q_n are adjacent if their strings differ in exactly one position. For example, in Q_4 the vertices* 1101 *and* 1001 *are adjacent but* 1100 *and* 1010 *are not adjacent.*

   a) *How many edges does Q_n have?*

   b) *Prove that Q_n is bipartite.*

   Solution:
   The n-cube Q_n is the graph whose set of vertices is $V = \{x_1 x_2 \cdots x_n : x_i \in \{0, 1\} \text{ for all } i = 1, 2, \ldots, n\}$. It is esay to see that $|V| = 2^n$. Two vertices u and v are adjacent if and only if their strings differ in exactly one position. This means that there exists exactly one index $j \in \{1, 2, \ldots, n\}$ such that $u = x_1 x_2 \cdots x_{j-1} x_j x_{j+1} \cdots x_n$ and $v = x_1 x_2 \cdots x_{j-1} \overline{x_j} x_{j+1} \cdots x_n$, where $\overline{x_j} = 1 - x_j$.

a) To compute the number of edges of Q_n we have to determine the value of the sum of the degrees of vertices of Q_n (recall that this sum is equal to $2|E|$). Consider a vertex $u = x_1x_2\cdots x_n$ of Q_n. This vertex is adjacent to the following vertices:

$$\begin{gathered}\overline{x_1}x_2\cdots x_n \\ x_1\overline{x_2}\cdots x_n \\ \vdots \\ x_1x_2\cdots\overline{x_n}\end{gathered}$$

where $\overline{x_j} = 1 - x_j$. This means that u is adjacent to exactly n vertices (i.e. $d(u) = n$) and, as a consequence, Q_n is a regular graph of degree n. This implies that the sum of the degrees of the vertices of Q_n is equal to $2^n \cdot n$, and so

$$2|E| = 2^n \cdot n \Rightarrow |E| = n \cdot 2^{n-1}.$$

b) The graph $Q_n = (V, E)$ is bipartite if the vertices of Q_n can be partitioned into two subsets A and B so that each edge of Q_n has one vertex in A and one vertex in B.
Consider the following sets:

$$A = \{x_1x_2\cdots x_n \in V : x_1 + x_2 + \cdots + x_n \text{ is even}\};$$

$$B = \{x_1x_2\cdots x_n \in V : x_1 + x_2 + \cdots + x_n \text{ is odd}\}.$$

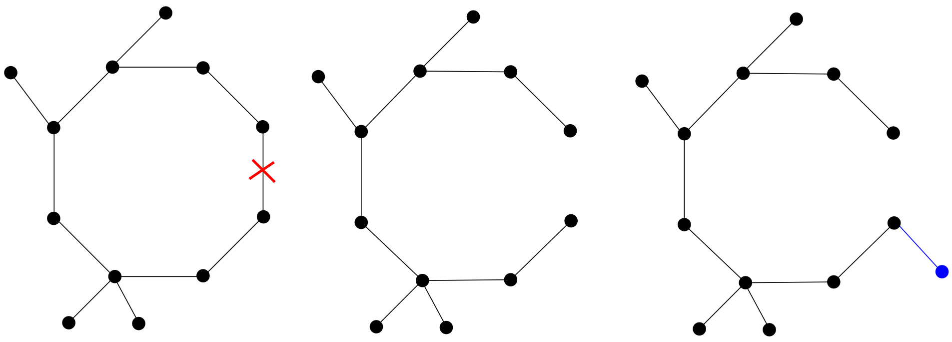
Clearly $A \cup B = V$ and $A \cap B = \emptyset$. It remains to prove that every edge of Q_n joins a vertex in A to a vertex in B.
Let $e = uv$ be an edge of Q_n. From the definition of Q_n we have that $u = x_1x_2\cdots x_{j-1}x_jx_{j+1}\cdots x_n$ and $v = x_1x_2\cdots x_{j-1}\overline{x_j}x_{j+1}\cdots x_n$.
Suppose that $x_j = 0$. Thus $\overline{x_j} = 1$.
If $x_1 + \cdots + x_{j-1} + 0 + x_{j+1} + \cdots + x_n$ is even, then $x_1 + \cdots + x_{j-1} + 1 + x_{j+1} + \cdots + x_n$ is odd. This means that if $u \in A$ then $v \in B$.
If $x_1 + \cdots + x_{j-1} + 0 + x_{j+1} + \cdots + x_n$ is odd, then $x_1 + \cdots + x_{j-1} + 1 + x_{j+1} + \cdots + x_n$ is even. This means that if $u \in B$ then $v \in A$.
We can repeat the same argument if $x_j = 1$. This proves that Q_n is bipartite.

4. *G is a connected graph with 20 edges. Find the maximum number of vertices that G can have.*

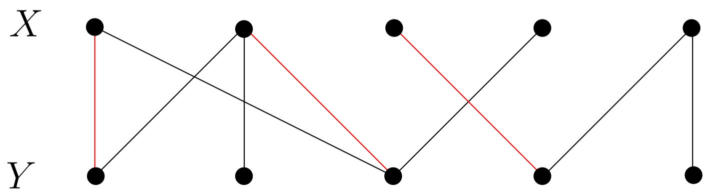
Solution:
If $G = (V, E)$ contains a cycle, we can transform G into a new graph with the same number of edges but with a greater number of vertices using the following procedure:

1. delete an edge of the cycle;
2. add a new edge that joins a vertex of G to a new vertex.

The graph $G' = (V', E')$ obtained in this way is connected with $|V'| = |V| + 1$ and $|E'| = |E|$.
As a consequence, the connected graph G with 20 edges and with the maximum number of vertices must not contain cycles; in other words G must be a tree. In this case, if we denote $n = |V|$, we obtain that $n - 1 = |E| = 20$, which implies that $n = 21$.



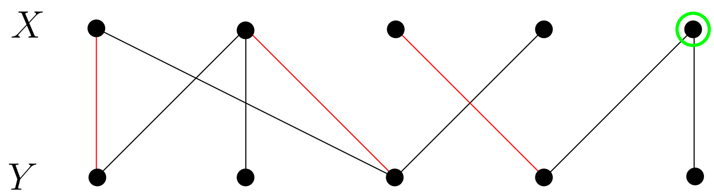
5. *Consider the following bipartite graph $G = (X \cup Y, E)$. Starting from the given matching M formed by red edges, construct a complete matching of G.*



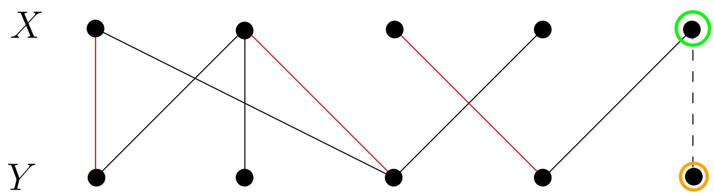
Solution:
A matching is a subset of the edge set E such that no two edges share a common vertex. If $|X| \leq |Y|$ and $|M| = |X|$, then M is called a complete matching. Recall that a bipartite graph admits a complete matching if and only if Hall's condition is satisfied: $|J(A)| \geq |A|$ for all $A \subseteq X$, where $J(A) = \{y \in Y | xy \in E \text{ for some } x \in A\}$. Note that Hall's condition is satisfied for the given graph.

In our case M is not a complete matching; thus we need to use the same procedure of the proof of Hall's theorem to construct a complete matching starting from M.

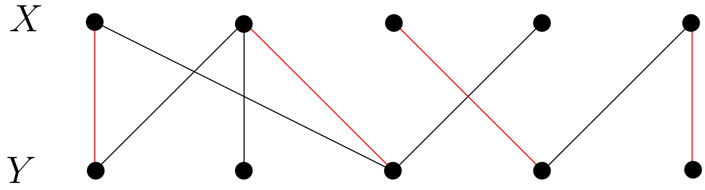
Consider the selected vertex of X which is unmatched.



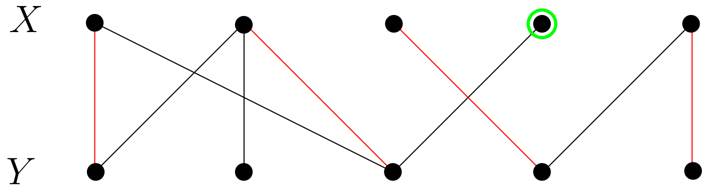
It is adjacent to a vertex of Y that is unmatched in M,



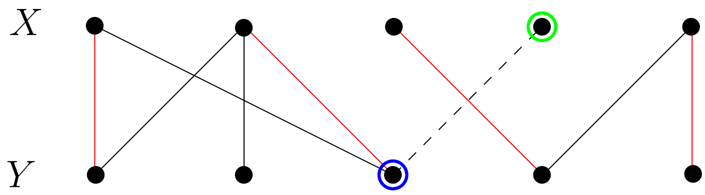
thus we can construct a new matching by adding the dashed edge to M, obtaining the matching M':

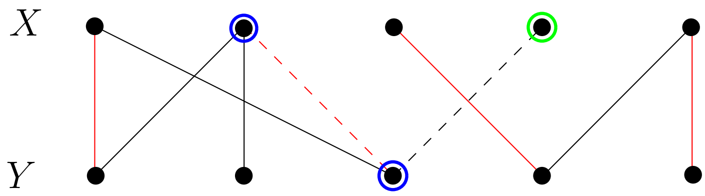


It remains one vertex in X that is unmatched:

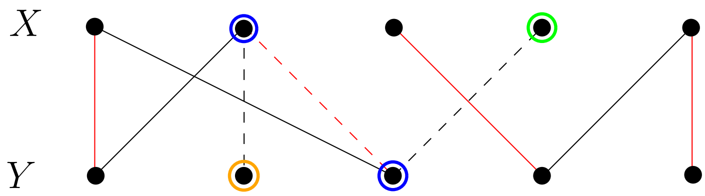


In this case the selected vertex is adjacent to only one vertex of Y which is already matched in M':

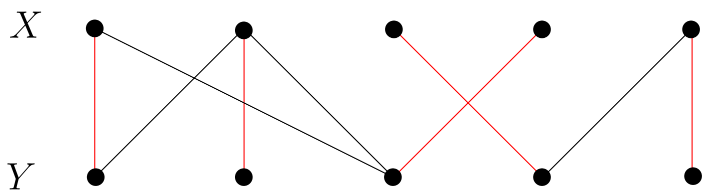




The blue vertex of X is adjacent to one vertex of Y that is unmatched:



We can see that in the dashed path P the first and last edges are not in M', while the middle edge is in M'. We construct a new matching M'' in which the first and last edges of P are in M'', while the middle edge is not in M'':



The matching M'' is a complete matching of G.